



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

We may observe that every algebraic identity, in which each term is of the second degree in so far as certain letters are concerned, may be given a geometric interpretation if each of such letters be used to represent a certain vector, and if the scalar but not the vector portion of the product be employed in the interpretation. That this is true follows from the fact that the non-commutative character of vector multiplication does not alter or affect the scalar portion of the product, if each term of such product contains either the product of two separate vectors or the square of some one vector; *i. e.*, if no term in the expanded form is of a degree higher or lower than the second in the letters used to designate vectors.

A solution similar to the foregoing may be employed in problem 377.

**384. Proposed by S. LEFSEHETZ, Clark University.**

Let  $ABC$  be a triangle,  $O$  a circle tangent to its three sides,  $T$  a variable tangent of  $O$ , which cuts the sides  $BC$ ,  $CA$ ,  $AB$  in  $a$ ,  $b$ ,  $c$ .  $Oa'$ ,  $Ob'$ ,  $Oc'$  the perpendiculars in  $O$  to  $Oa$ ,  $Ob$ ,  $Oc$ , cutting, respectively,  $T$  in points  $a'$ ,  $b'$ ,  $c'$ . Prove that  $Aa'$ ,  $Bb'$ ,  $Cc'$  meet in a point  $t$ , and find the locus of  $t$  when  $T$  varies. Purely geometrical proofs wanted.

No solution of this problem has been received.

**385<sup>2</sup>. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.**

Given a triangle  $ABC$ , find the radius of a circle touching two of its sides and a line parallel to the third, at a distance  $d=u+2r$ .

**Solution by A. H. HOLMES, Brunswick, Maine.**

Let  $a$ ,  $b$ , and  $c$  be the sides of the given triangle,  $c$  the base. Then  $h=\text{altitude of the triangle}=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2c}$ , and  $R=\text{radius of the inscribed circle}=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)]}}{2(a+b+c)}$ .

Put  $r=\text{radius of circle touching } a \text{ and } b \text{ and a line parallel to } c \text{ at a distance from } c, 2r+u$ . Then  $h:R=h:(2r+u):r$ .

$$\therefore r=\frac{(h-u)R}{h+2R}.$$

Putting for  $h$  and  $R$  their values in terms of  $a$ ,  $b$ , and  $c$ , we have,

$$r=\frac{\sqrt{[4a^2c^2-(a^2-b^2+c^2)^2]}-u}{a+b+3c}.$$

**CALCULUS.**

**306. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.**

Express in elliptic integrals:  $A_\theta = \int_0^\theta \frac{dx}{\sqrt{(1-x^4)}} ; 0 < \theta < 1$ .

I Solution by WALTER D. LAMBERT, A. M., University of Pennsylvania, Philadelphia, Pa.

Correcting the inequality to read  $0 < \theta < 1$ , we find (by using the substitution  $x = \cos \phi$ , and by calling  $\alpha = \cos^{-1} \theta$ ) that

$$\begin{aligned} A_\theta &= \int_0^\theta \frac{dx}{\sqrt{[(1-x^2)(1+x^2)]}} = - \int_{\frac{\pi}{2}}^\alpha \frac{\sin \phi \, d\phi}{\sqrt{[(1-\cos^2 \phi)(1+\cos^2 \phi)]}} \\ &= \int_{\alpha}^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{[2-\sin^2 \phi]}} = \frac{1}{\sqrt{2}} \int_{\alpha}^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{[1-\frac{1}{2}\sin^2 \phi]}} \\ &= \frac{1}{\sqrt{2}} \left[ F\left(\frac{\pi}{2}\right) - F(\alpha) \right] \text{ modulus } \frac{1}{\sqrt{2}} \end{aligned}$$

where  $F\left(\frac{\pi}{2}\right)$  and  $F(\alpha)$  are Legendre's elliptic integrals of the first kind.

As a numerical example take  $\theta = \frac{1}{2}$ .  $\therefore \alpha = \frac{1}{3}\pi$ .

$A_\theta = \frac{1}{\sqrt{2}} [1.8451 - 1.1424] = 0.5032$ , using the four-place "funktionentafeln" of Jahnke and Emde. As a verification, expand  $\frac{1}{\sqrt{(1-x^4)}}$  by the binomial theorem and integrate:

$$A_\theta = \int_0^\theta \frac{dx}{\sqrt{[1-x^4]}} = \int_0^\theta (1 + \frac{1}{2}x^4 + \frac{3}{8}x^8 + \frac{5}{16}x^{12} \dots) dx = \theta + \frac{\theta^5}{10} + \frac{\theta^9}{24} + \frac{5}{208}\theta^{13}.$$

On substituting  $\theta = \frac{1}{2}$  we get .5032 for  $A_\theta$  as before.

II. Solution by the PROPOSER.

Let  $x = \tan \omega$ . Then  $dx = \frac{d\omega}{\cos^2 \omega}$ . Substituting these values in the integral expression, we have

$$\begin{aligned} A_\theta &= \int_0^\omega \frac{d\omega}{\cos^2 \omega \sqrt{[(1-\tan^2 \omega)(1+\tan^2 \omega)]}} \\ &= \int_0^\omega \frac{d\omega}{\sqrt{[\cos^2 \omega - \sin^2 \omega]}} = \int_0^\omega \frac{d\omega}{\sqrt{[1-2\sin^2 \omega]}}, \end{aligned}$$

where  $\tan \omega = \theta$  is used for the upper limit.

Now let  $\sqrt{2}\sin \omega = \sin \phi$ ; then  $1-2\sin^2 \omega = \cos^2 \phi$  and

$$d\omega = \frac{\cos\phi \, d\phi}{\sqrt{2} \sqrt{[1 - \frac{1}{2}\sin^2\phi]}}.$$

Whence  $A_\theta = \frac{1}{2}\sqrt{2} \int_0^\phi \frac{d\omega}{\sqrt{[1 - \frac{1}{2}\sin^2\phi]}} = \frac{1}{2}\sqrt{[2]} F(\frac{1}{2}\sqrt{2}, \phi).$

The amplitude  $\phi$  is determined from  $\theta = \tan \omega$ ,  $\cos^2 \omega = \frac{1}{1+\theta^2}$ ,  $\sin^2 \omega = \frac{\theta^2}{1+\theta^2}$ ,  $\sin^2 \phi = \frac{2\theta^2}{1+\theta^2}$ ,  $\cos^2 \phi = \frac{1-\theta^2}{1+\theta^2}$ . Hence,  $\tan^2 \phi = \frac{2\theta^2}{1-\theta^2}$ .

Referring to problem 303,

$$A = \int_0^1 \frac{dx}{\sqrt{[1-x^4]}} = \frac{\sqrt{\pi} \Gamma(\frac{1}{4})}{4 \Gamma(\frac{3}{4})},$$

the well known formula  $\Gamma(\theta)\Gamma(1-\theta) = \pi/\sin \pi \theta$ , gives in our case  $\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi \sqrt{2}$ . Whence,  $\Gamma(\frac{3}{4}) = \frac{\pi \sqrt{2}}{\Gamma(\frac{1}{4})}$ , and therefore,  $A = \frac{[\Gamma(\frac{1}{4})]^2}{4 \sqrt{[2\pi]}}.$

This combined with the result in above,  $A = \frac{1}{2}\sqrt{[2]} F'(\frac{1}{2}\sqrt{2})$  yields  $\Gamma(\frac{1}{4}) = 2\sqrt{[\pi]} \{F'(\frac{1}{2}\sqrt{2})\}^{\frac{1}{2}} = 3.62561, *0.5593811.$

And similarly,  $\Gamma(\frac{3}{4}) = \frac{1}{2}\sqrt{[2]} \sqrt[4]{[\pi^3]} \{F'(\frac{1}{2}\sqrt{2})\}^{-\frac{1}{2}} = 1.225416, *0.0882838.$

These expressions for  $\Gamma(\frac{1}{4})$  and  $\Gamma(\frac{3}{4})$  in Legendre's  $F$ -functions are to my mind by far the most important consequences of evaluating integral  $A$  in gamma-functions. Without this evaluation  $\Gamma(\frac{1}{4})$  and  $\Gamma(\frac{3}{4})$  can be determined only by computing their natural logarithms by inconvenient series.

Also solved by V. M. Spunar, C. N. Schmall, and J. Scheffer.

## MECHANICS.

253. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

$R_1$  and  $R_2$  are ranges on a horizontal plane of particles projected with given velocity from  $A$  on the plane to pass through  $B$ . Show that  $a(R_1 + R_2) - R_1 R_2 = \frac{a^4}{c^2}$ , where  $c = AB$  and  $a$  is the horizontal projection of  $AB$ .

### III. Solution by the PROPOSER.

If  $\alpha$ ,  $\alpha_1$  be angles of projection and  $\beta$  the angle  $AB$  makes with the horizontal, and  $v$  the velocity of projection, then  $\alpha_1 = \frac{1}{2}\pi - (\alpha - \beta)$ .

And  $\cos \beta = a/c$ ;  $R_1 \cdot g = a^2 \sin 2\alpha$ ;  $R_2 \cdot g = a^2 \sin 2(\alpha - \beta)$ .

$$AB = c = \frac{2v^2}{g} \frac{\cos \alpha \sin(\alpha - \beta)}{\cos \beta} = \frac{2v^2 c^2}{g a^2} \sin(\alpha - \beta) \cos \alpha.$$